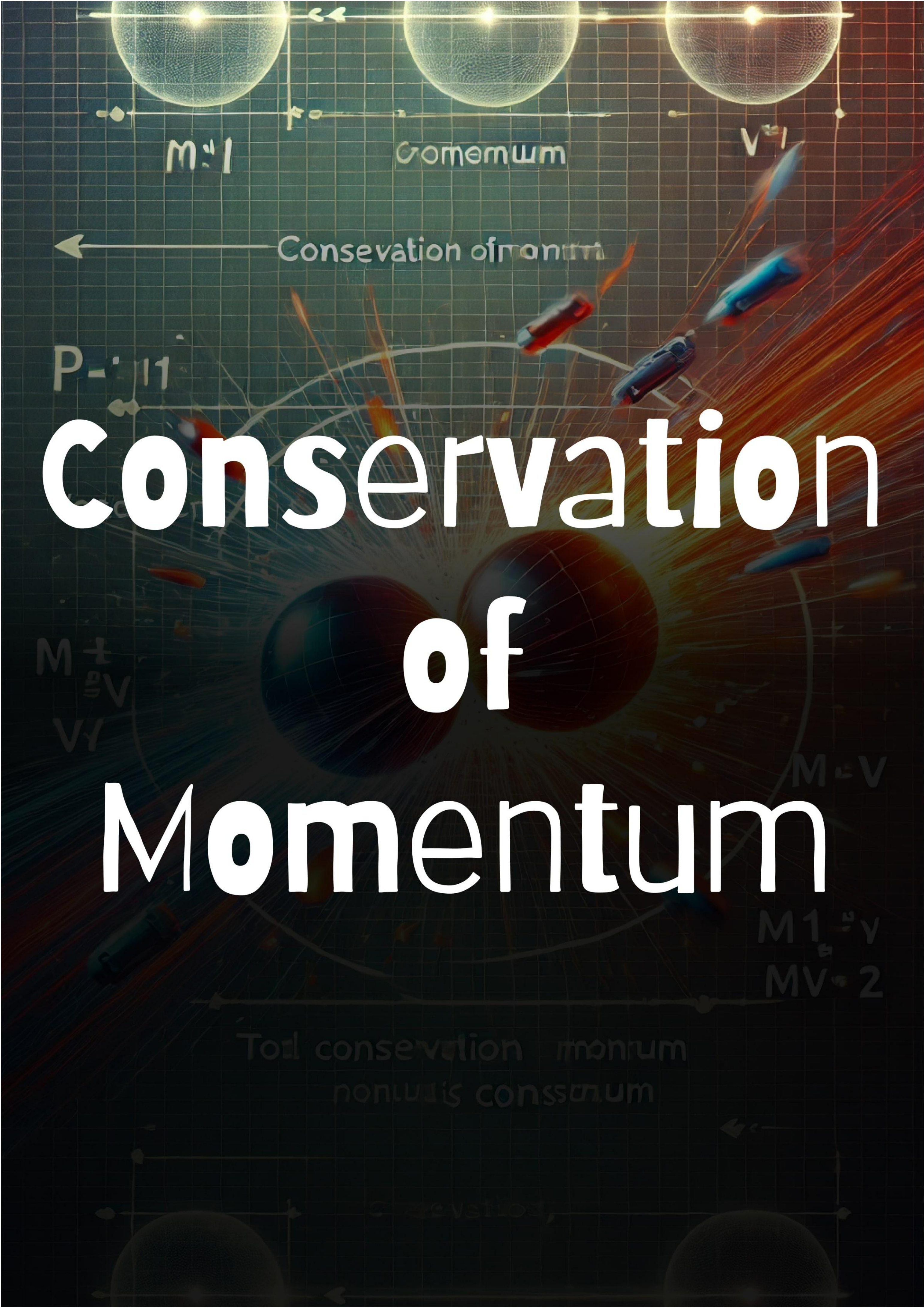


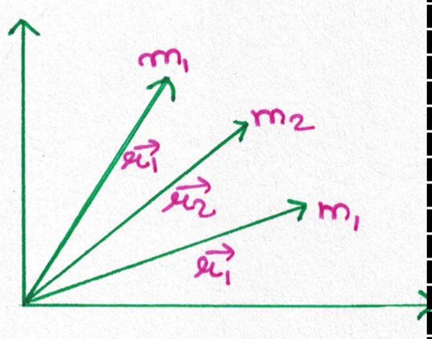
Conservation of Momentum

Conservation of momentum

CENTER OF MASS

It is average location of mass in space.

Mathematically, $\vec{r}_{cm} = \dfrac{\sum m_i \cdot \vec{r}_i}{\sum m_i}$

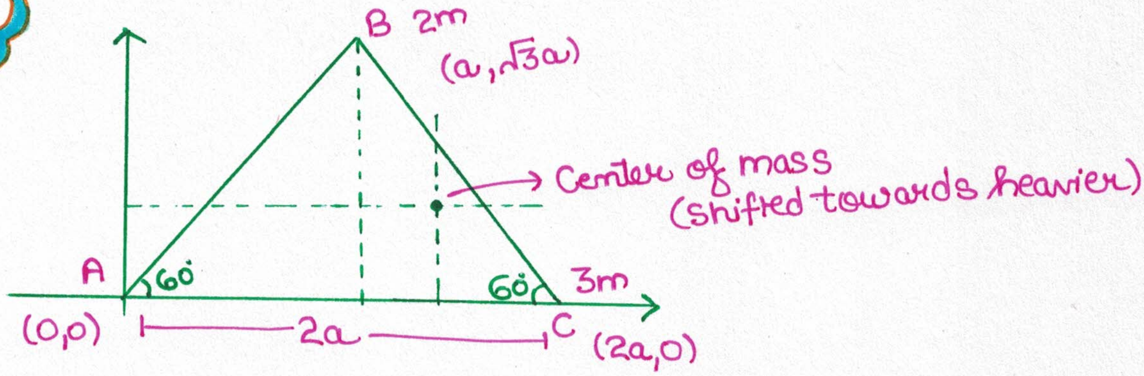


$$x_{cm} = \frac{\sum m_i x_i}{\sum m_i}$$

$$y_{cm} = \frac{1}{M} \sum m_i y_i$$

$$z_{cm} = \frac{1}{M} \sum m_i z_i$$

PRACTICE TIME



$$x_{cm} = \frac{m \times 0 + 2m \times a + 3m \times 2a}{6m}$$

$$x_{cm} = \frac{4a}{3}$$

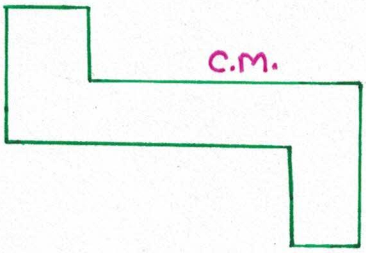
$$y_{cm} = \frac{m \times 0 + 2m \times \sqrt{3}\,a + 3m \times 0}{6m}$$

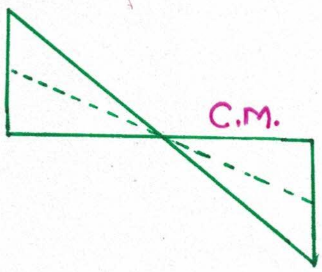
$$= \frac{\sqrt{3}}{3} a$$

LOCATING CENTER OF MASS BY SYMMETRY

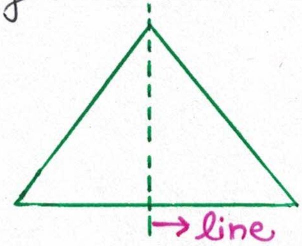
SYMMETRY ABOUT A POINT :

1.) If in any homogenous object, there exists a point of symmetry (180° opposite points are equally beautiful) then that point is itself the centre of mass.

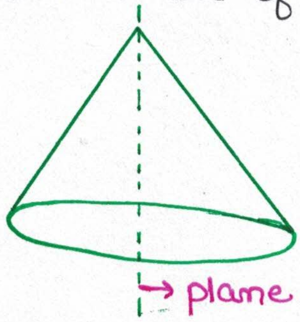




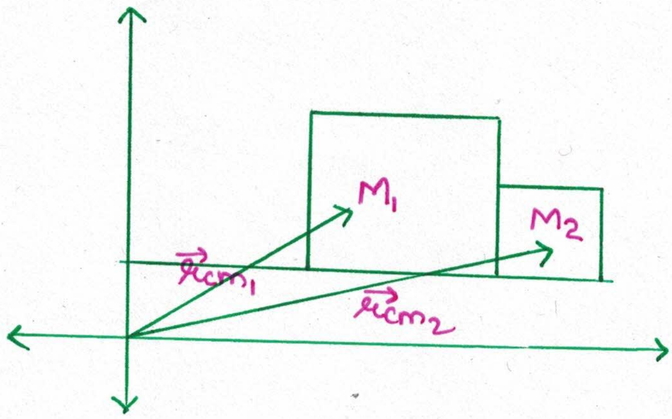
2.) In two-dimensional homogenous objects, if there exists a line of symmetry, then centre of mass lies on it,



3.) For 3 dimensional homogenous objects, if there exists a plane of symmetry then the centre of mass lies on it.

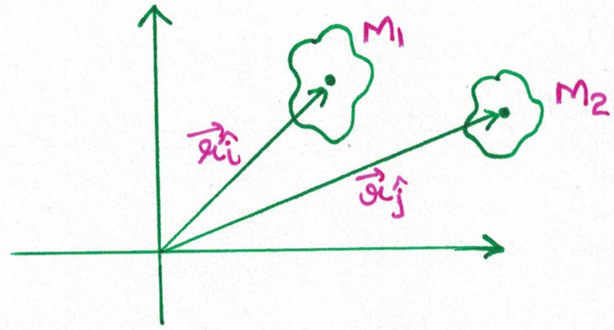


FINDING C.M. BY COMPOSITE BODY METHOD



$$\vec{r}_{cm} = \frac{M_1\vec{r}_{cm_1} + M_2\vec{r}_{cm_2}}{M_1 + M_2}$$

PROOF



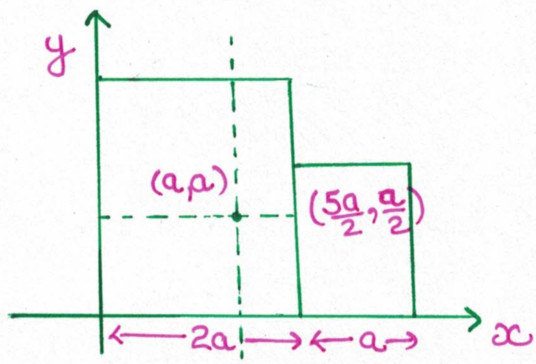
$$\vec{r}_{cm_1} = \frac{\sum m_i \cdot \vec{r}_i}{M_1}$$

$$\vec{r}_{cm_2} = \frac{\sum m_j \cdot \vec{r}_j}{M_2}$$

$$\vec{r}_{cm} = \frac{\sum m_i \vec{r}_i + \sum m_j \vec{r}_j}{M_1 + M_2}$$

$$\vec{r}_{cm} = \frac{M_1 \vec{r}_{cm_1} + M_2 \vec{r}_{cm_2}}{M_1 + M_2}$$

Que:)

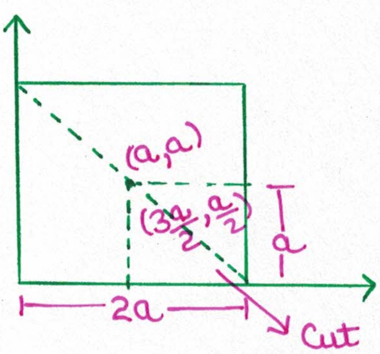


$$x_{cm} = \frac{4Ma + M(5a/2)}{5M} = \frac{13}{10}a$$

$$y_{cm} = \frac{4Ma + M \times a/2}{5M} = \frac{9a}{10}$$

USING NEGATIVE MASS IN COMPOSITE BODY

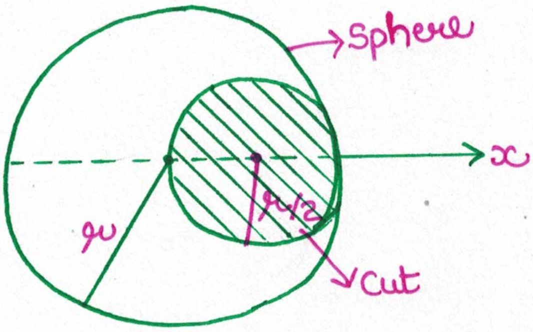
Que:)



$$x_{cm} = \frac{4Ma - M \times \frac{3a}{2}}{4M - M} = \frac{5a}{6}$$

$$y_{cm} = \frac{4Ma - M\frac{a}{2}}{4M - M} = \frac{7a}{6}$$

Que.)



For sphere, $x_{cm} = \dfrac{-\frac{R}{2}\cancel{M}}{7\cancel{M}} = -\dfrac{R}{14}$

For circle, $x_{cm} = \dfrac{-\frac{R}{2}\cancel{M}}{3\cancel{M}} = -\dfrac{R}{6}$

FINDING C.M. BY INTEGRATION

$$\vec{r}_{cm} = \frac{\sum M_i \vec{r}_{cmi}}{\sum M_i}$$

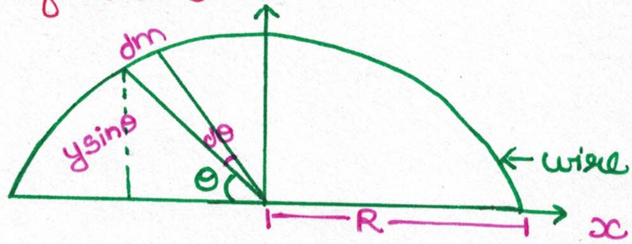
$$\therefore \vec{r}_{cm} = \frac{1}{M}\int \vec{r}\,.\,dm$$

Here, r is the coordinate of the center of the mass of body.

$$x_{cm} = \frac{1}{M}\int x\,dm$$

$$y_{cm} = \frac{1}{M}\int y\,dm$$

Que.) Locate the center of mass of a thin homogenous semi-circular wire of radius R.



$$y_{cm} = \frac{1}{M}\int y\,dm$$

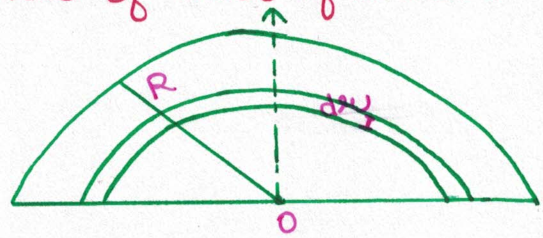
$$dm = \frac{M}{\pi}d\theta \quad , \quad y \rightarrow R\sin\theta$$

$$y_{cm} = \frac{1}{M}\int R\sin\theta \,.\, \frac{M}{\pi}d\theta$$

$$= \frac{R}{\pi}\int_0^{\pi} \sin\theta\,.\,d\theta$$

$$= -\frac{R}{\pi}\Big[\cos\theta\Big]_0^{\pi} = \frac{2R}{\pi}$$

Que:) Locate the centre of mass of a semi-circular lamina.



$$y_{cm} = \frac{1}{M}\int y\,dm$$

$$dm = \frac{M}{\pi R^2/2} \times \pi r \times dr = \frac{2M}{R^2} r\,dr$$

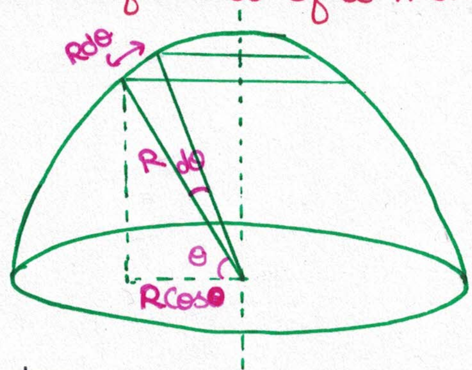
$$y = \frac{2r}{\pi}$$

$$y_{cm} = \frac{1}{M}\int \frac{2r}{\pi}\cdot\frac{2M}{R^2}\, r\,dr$$

$$= \frac{4}{\pi R^2}\int_0^R r^2\,dr$$

$$y_{cm} = \frac{4R}{3\pi}$$

Que:) Locate the centre of mass of a hemispherical cap.



$$y_{cm} = \frac{1}{M}\int y\,dm$$

$$dm = \frac{M}{2\pi R^2} \times 2\pi R\cos\theta . R\,d\theta$$

$$= M\cos\theta . d\theta$$

$$y = R\sin\theta$$

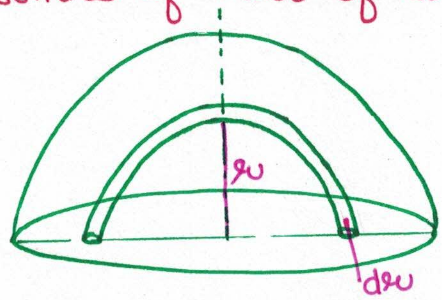
$$y_{cm} = \frac{1}{M}\int R\sin\theta . M\cos\theta . d\theta$$

$$= R\int_0^{\pi/2} \frac{\sin 2\theta}{2}$$

$$= -\frac{R}{4}\left[\cos 2\theta\right]_0^{\pi/2}$$

$$y_{cm} = \frac{R}{2}$$

Que.) Locate the centre of mass of a solid hemisphere.



$$y_{cm} = \frac{1}{M}\int y\,dm$$

$$dm = \frac{M}{\cancel{2}/3\,\pi R^3} \times dr \times \cancel{2}\pi r^2$$

$$= \frac{3r^2 M}{R^3}\cdot dr$$

$$y = \frac{r}{2}$$

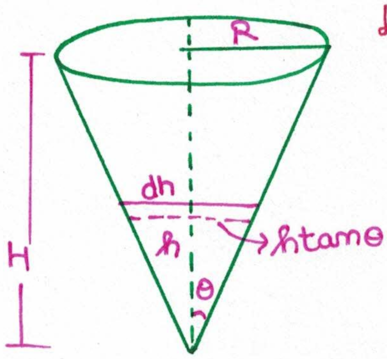
$$y_{cm} = \frac{1}{M}\int \frac{r}{2} \times \frac{3r^2 M}{R^3}\cdot dr$$

$$= -\frac{3}{2R^3}\int_0^R r^3\cdot dr$$

$$= \frac{3}{2R^3}\left[\frac{r^4}{4}\right]_0^R$$

$$y_{cm} = \frac{3R}{8}$$

Que.) Locate the C.M. of a solid cone.



$$\tan\theta = \frac{R}{H}$$

$$y_{cm} = \frac{1}{M}\int y\,dm$$

$$dm = \frac{M}{\frac{1}{3}\cancel{\pi} R^2 H} \times \cancel{\pi}\, h^2 \tan^2\theta\cdot dh$$

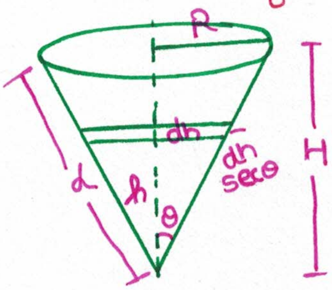
$$dm = \frac{3M}{H^3}h^2\cdot dh$$

$$y = h$$

$$y_{cm} = \frac{1}{M}\int_0^H h \times \frac{3M}{H^3} \times h^2\cdot dh$$

$$= \frac{3}{H^3}\int_0^H h^3\cdot dh = \frac{3H}{4} \quad \text{(from the tip)}$$

Que.) Locate the C.M. of hollow cone.



$$\sec\theta = \frac{l}{H}$$

$$\tan\theta = \frac{R}{H}$$

$$y_{cm} = \frac{1}{M}\int y \cdot dm$$

$$dm = \frac{M}{\pi R l} \times dh \sec\theta \times 2\pi h\tan\theta$$

$$= \frac{2M h \times l \times R}{l \times H \times H} \cdot dh$$

$$dm = \frac{2M\cancel{R}h}{H^2\cancel{R}} \cdot dh = \frac{2M}{H^2} h \cdot dh$$

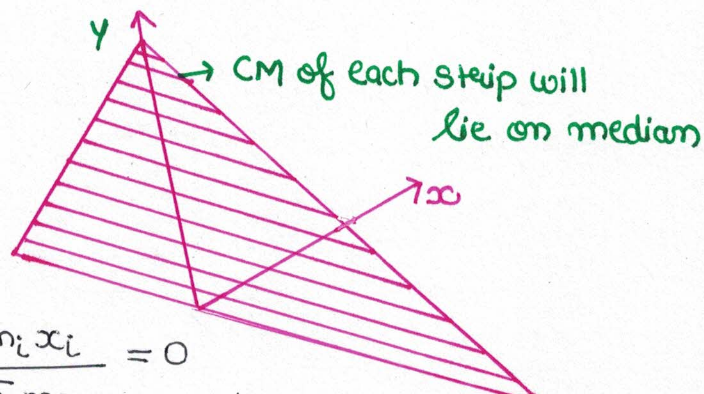
$$y = h$$

$$y_{cm} = \frac{1}{M}\int h \cdot \frac{2M}{H^2} h \cdot dh$$

$$= \frac{2}{H^2}\int_0^H h^2 \cdot dh$$

$$y_{cm} = \frac{2}{3}H \quad \text{(from the tip)}$$

Que.) Show that centre of mass of a triangular lamina lies on the intersection of medians.

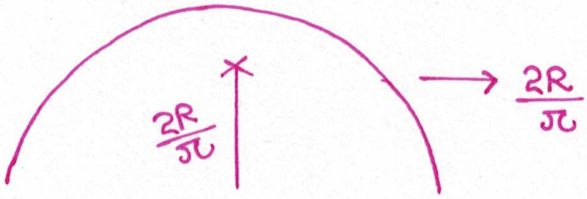


$$x_{cm} = \frac{\sum m_i x_i}{\sum m_i} = 0$$

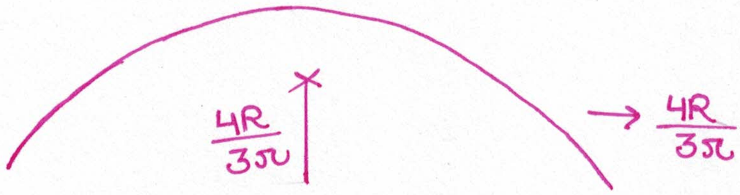
$\therefore$ CM of triangle is lying on each median.

SUMMARY OF RESULTS

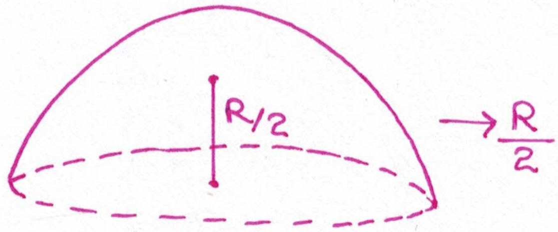
1.) Semi- Circular wire



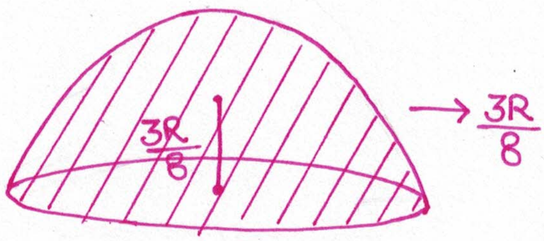
2.) Semi- circular lamina



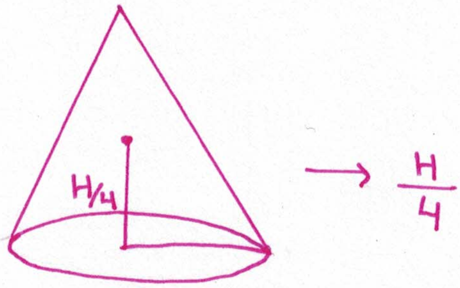
3.) Hemispherical shell



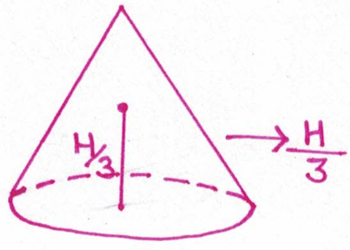
4.) Solid Hemisphere



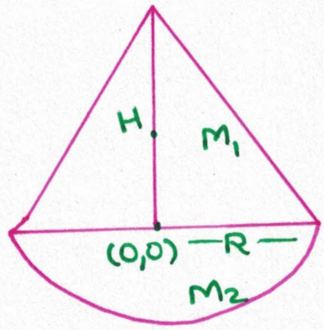
5.) Solid Cone



6.) Hollow Cone



Que:) Find max. ratio of H/R so that this toy works.



$$x_{cm} = \frac{M_1 \frac{H}{3} - M_2 \frac{R}{2}}{M_1 + M_2} = \frac{\sigma \pi R l \times \frac{H}{3} - \sigma 2\pi R^2 \left(\frac{R}{2}\right)}{\sigma \pi R l + \sigma \times (2\pi R^2)} = 0$$

$\therefore$ Numerator $= 0$

$$\& \quad \frac{lH}{3} = R^2$$

$$\sqrt{H^2+R^2} \cdot \frac{H}{3} = R^2$$

$$\sqrt{1+\left(\frac{H}{R}\right)^2} \cdot \frac{H}{R} = 3$$

Putting $\frac{H}{R} = x$

$$\sqrt{1+x^2} \times x = 3$$

$$(1+x^2) = \frac{9}{x^2}$$

$$1+y = \frac{9}{y}$$

$$\sqrt{\frac{\sqrt{37}-1}{2}} = \frac{H}{R}$$

MOTION OF CENTRE OF MASS

$$\vec{r}_{cm} = \frac{1}{M} \sum m_i \vec{r}_i$$

$$\frac{d\vec{r}_{cm}}{dt} = \frac{1}{M} \sum m_i \frac{d\vec{r}_i}{dt}$$

$$\boxed{\vec{v}_{cm} = \frac{1}{M} \sum m_i \vec{v}_i} \quad — (1)$$

Momentum of system of particles

$$P = M \vec{v}_{cm} \quad — (2)$$

ACCELERATION OF CENTRE OF MASS

$$\vec{a}_{cm} = \frac{d\vec{v}_{cm}}{dt} = \frac{1}{M}\sum m_i \frac{d\vec{v}_i}{dt}$$

$$\boxed{\vec{a}_{cm} = \frac{1}{M}\sum m_i \vec{a}_i} \quad \text{— (3)}$$

NET EXTERNAL FORCE ON SYSTEM OF PARTICLES

$$\sum F_i = M\vec{a}_{cm}$$

$$\boxed{\vec{F}_{ext} = M\vec{a}_{cm}} \quad \text{— (4)}$$

Since internal forces cancel out

Net external force is mass times acceleration of centre of mass.

LAW OF CONSERVATION OF MOMENTUM (COM)

If the net external force on the system is zero, then its momentum remains conserved.

Mathematically,

$$\text{If } \vec{F}_{ext} = 0$$

$$\text{then } \vec{P} = \text{constant}$$

Since this is a vector condition, it leads to the following 3 component conditions :

(1) If $\vec{F}_{ext_x} = 0$ then $\vec{P}_x = C$

(2) If $\vec{F}_{ext_y} = 0$ then $\vec{P}_y = C$

(3) If $\vec{F}_{ext_z} = 0$ then $\vec{P}_z = C$

This can be shown as follows:

If $\vec{F}_{ext} = 0$

$\vec{a}_{cm} = 0$ (by (4))

$\vec{v}_{cm} = C$

$M\vec{v}_{cm} = C$ (M is a constant)

$\vec{P} = C$ (from (2))

CONSERVATION OF CENTRE OF MASS (CCOM)

If net external force on a system is zero and in addition the initial velocity of centre of mass is also zero, then the position of centre of mass remains fixed.

Mathematically,

$$\text{If } \vec{F}_{ext} = 0$$
$$\& \ \vec{v}_{cm} = 0$$
$$\vec{r}_{cm} = C$$
$$\text{or } \sum m_i \Delta\vec{r}_i = 0$$

This can be shown as follows

$$\text{If } \vec{F}_{ext} = 0$$
$$\vec{v}_{cm} = C \qquad (\because \text{COM})$$

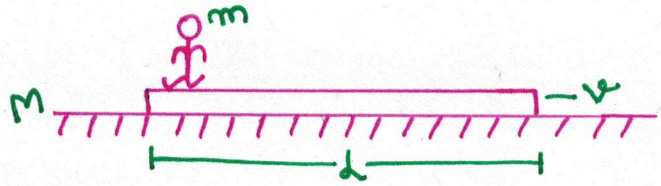
additionally if

$$\vec{v}_{cm} = 0$$
$$\vec{v}_{cm} \text{ is constantly zero}$$
$$\frac{d\vec{r}_{cm}}{dt} = 0$$
$$\vec{r}_{cm} = C$$
$$\therefore \Delta\vec{r}_{cm} = 0$$
$$\vec{r}_{cm} = \frac{\sum m_i \vec{r}_i}{M}$$
$$\Delta\vec{r}_{cm} = \frac{\sum m_i \vec{r}_i}{M} = 0$$
$$\boxed{\sum m_i \Delta\vec{r}_i = 0}$$

It is also a vector condition

Que) A man of mass 'm' rests on a stationary planck of mass 'M'. The ground is perfectly smooth. Now the man begins to move with a velocity 'u' relative to the planck. Find
(i) Speed of the planck.
(ii) displacement of the man in the ground frame when he reaches the other end of the planck.

let velocity of planck be v in +ve x direction



Using COM

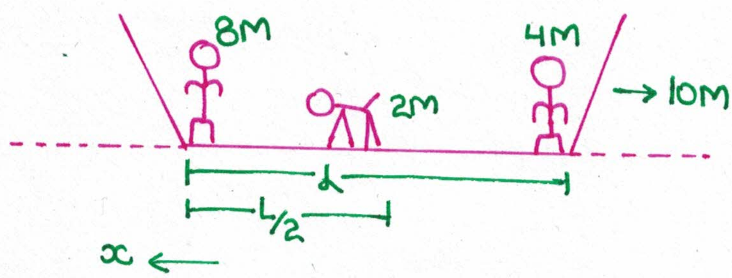
$$0 = m(u+v) + Mv$$
$$v = -\frac{mu}{M+m}$$

Let x be the displacement of planck in +ve x direction

<u>CCOM</u>

$$m(x+L) + Mx = 0$$
$$x = -\frac{mL}{M+m}$$
$$x+L = \frac{ML}{M+m} \quad \text{(displacement of man)}$$

<u>Que.)</u>



CCOM

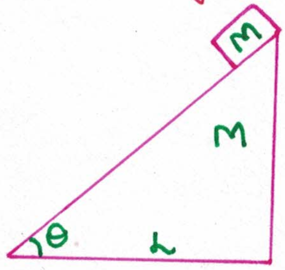
$$10m(x) + 4m(L+x) + 2m\left(\tfrac{L}{2}+x\right) + 8m(x) = 0$$
$$5mL + 24mx = 0$$
$$x = -\frac{5}{24}L$$

<u>Que.)</u> In the shown system $\mu = 0$ everywhere & the system is released from rest. Find

(i) Velocity of block A relative to the wedge when it releases the bottom (v)

(ii) Velocity of wedge relative to ground at that instant.

(iii) displacement of the wedge relative to ground at that instant.



<u>COME</u>

$$L.P.E. = G.K.E.$$

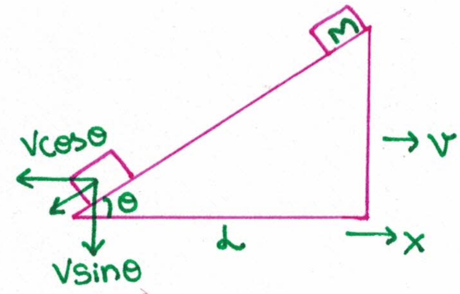
$$mg(L\tan\theta) = \frac{1}{2}m\left[(v\cos\theta - V)^2 + v^2\sin^2\theta\right] + \frac{1}{2}MV^2$$

COM (in x-direction)

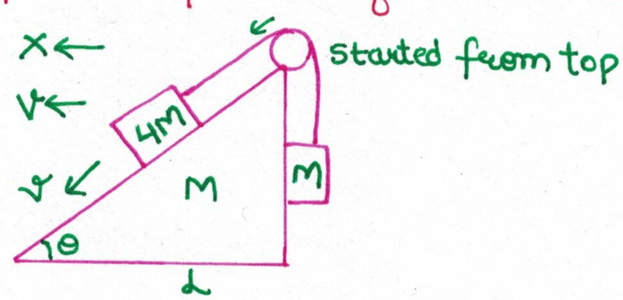
$$M(v\cos\theta - v) + Mv = 0$$

CCOM (in x-direction)

$$M(\ell - x) + Mx = 0$$



Que.) Repeat the previous problem for the shown case.



COME $\quad \ell.P.E. = G.K.E$

$$4Mg(\ell\tan\theta) - Mg\ell\sec\theta = \frac{1}{2}Mv^2 + \frac{1}{2}(4M)(v\cos\theta + v)^2 + (v^2\sin^2\theta) + \frac{1}{2}M(v^2 + u^2)$$

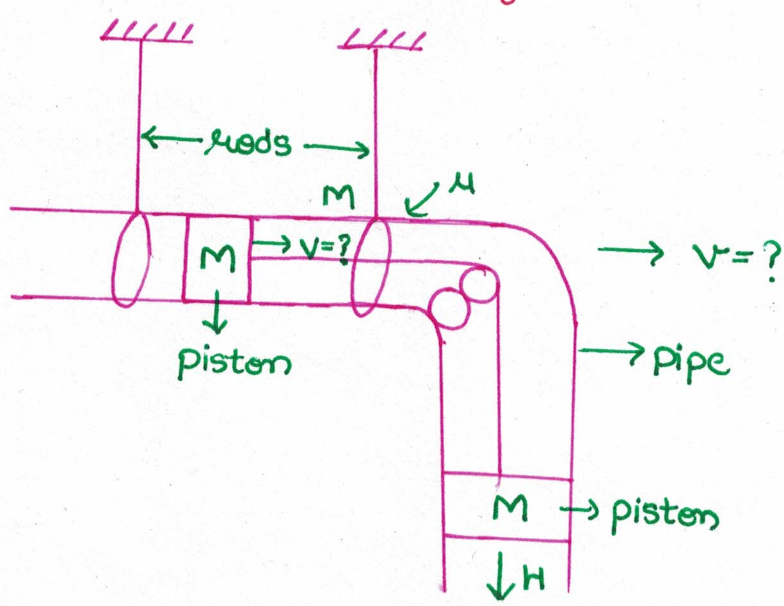
COM

$$Mv + Mv + 4M(v + v\cos\theta) = 0$$

CCOM

$$Mx + Mx + 4M(\ell + x) = 0$$

Que.) Repeat the previous problem for the shown fig.



COME

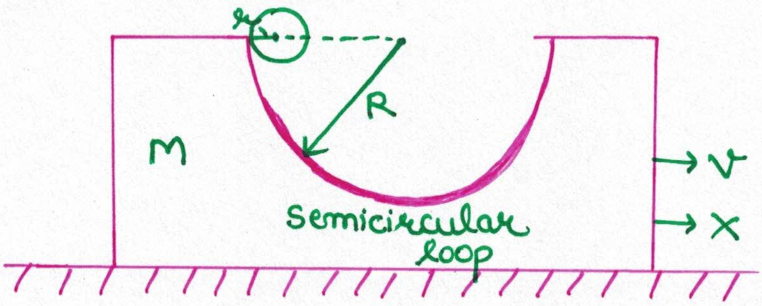
$\ell.P.E. = G.K.E.$

$$MgH = \frac{1}{2}M(v^2 + v^2) + \frac{1}{2}Mv^2 + \frac{1}{2}M(v + v)^2$$

$COM_x \quad M(v+v) + Mv + Mv = 0$

Que.) Find the velocity of cylinder when it reaches the bottom-most point.

(ii) Find velocity of block.



all surfaces are frictionless

COME $\quad \Delta PE = \Delta KE$

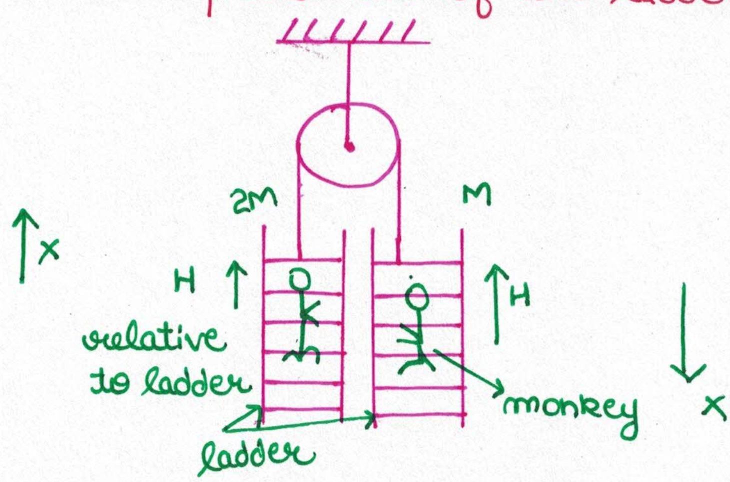
$$mg(R-r) = \frac{1}{2}m(V+v)^2 + \frac{1}{2}Mv^2$$

$COM_x \quad m(V+v) + MV = 0$

$CCOM_x$ (when cylinder reaches other extreme)

$$Mx + 2m(R-r) = 0$$

Que.) Find the displacement of the ladder relative to ground.



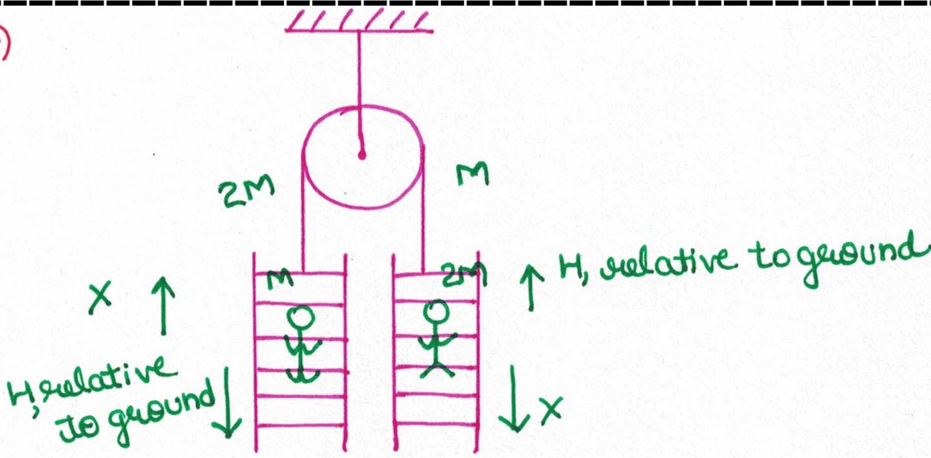
displacement of C.M. on both sides is same

$$\frac{M(x+H)+2Mx}{3M} = \frac{2M(H-x)+M(-x)}{3M}$$

$$3Mx + MH = 2MH - 3Mx$$

$$x = \frac{H}{6}$$

Que.)



$$\frac{M(-H)+2MX}{3\cancel{M}} = \frac{2MH+M(-X)}{3\cancel{M}}$$

$$-MH+2MX = 2MH-MX$$

$$H = X$$

LINEAR IMPULSE MOMENTUM THEOREM (LIM)

IMPULSE

The time integral of force is called the impulse. Mathematically, if a force $\vec{F}$ acts from time $t=t_1$ to $t=t_2$, then its corresponding impulse vector for the interval is given by

$$\vec{J} = \int_{t_1}^{t_2} \vec{F}.dt$$

Graphically, impulse is given by area under the F-t graph.

LINEAR IMPULSE MOMENTUM

Net impulse on any system is equal to the change in linear momentum of that system

$$\vec{F} = \frac{d\vec{P}}{dt}$$

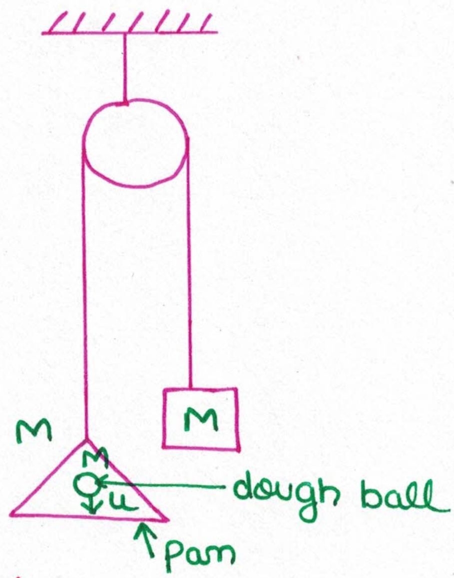
$$\int d\vec{P} = \int \vec{F}.dt$$

$$\vec{J} = \Delta\vec{P}$$

NOTE: While analysing the sudden phenomenon such as an explosion, collision or sudden tightening of a spring we neglect the impulse of relatively smaller forces such as Mg and spring forces. (When considering for a short window of time.

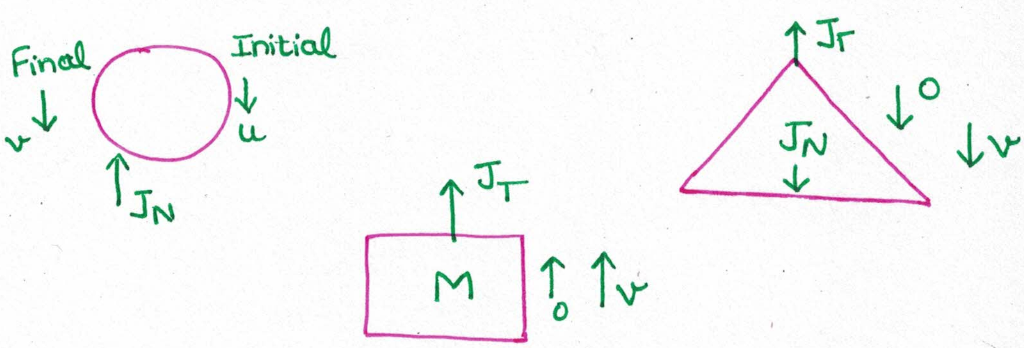
To apply the impulse momentum theory, we draw FID (Force impulse diagram) just as we draw FBD. While drawing the FID for a sudden phenomenon, don't show the small impulses.

Que.)



A dough ball of Mass M is thrown on a counter balanced pan with a speed u as shown
(i) Draw the FID of the pan, ball and block.
(ii) Write the LIM eq^n for all the 3 objects.
(iii) Solve for impulsive tension and impulsive normal reac^n. (J_T & J_N)

(i)



(ii) For dough ball, (LIM)

$$J_N = -Mv - (-Mu) \quad —①$$

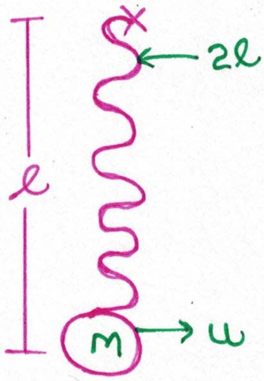
$(LIM)_{pan}$ $\quad J_N - J_T = Mv \quad$ — ②

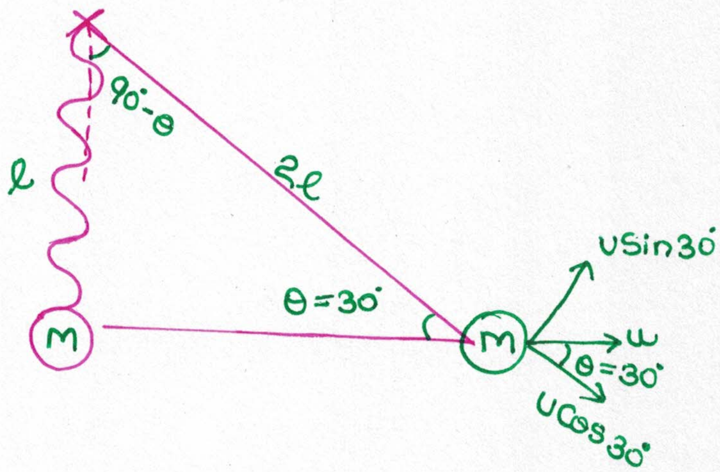
$(LIM)_{block}$ $\quad J_T = Mv \quad$ — ③

$$J_N = Mu - Mv$$
$$J_T = \frac{Mu - Mv}{2}$$
$$2Mv = Mu - Mv$$
$$v = \frac{u}{3}$$

$J_T = \frac{Mu}{3}$ and $J_N = \frac{2Mu}{3}$

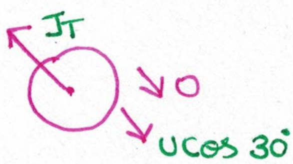
Que:) Find impulse tension & post impulsive velocity of the ball.





String will be taut
∴ velocity along the string = 0
(velocities of both end of string is zero.)

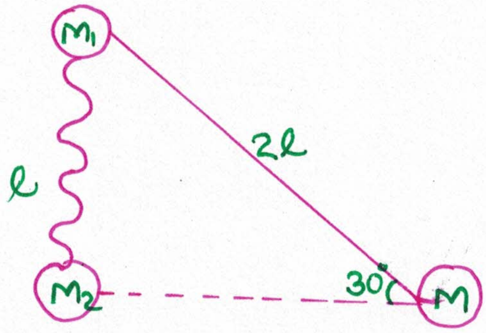
FID

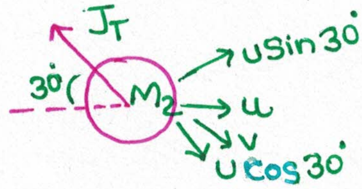


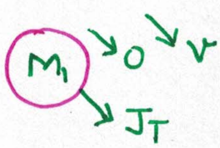
$J_T = +mu\cos 30°$ and $v = u\sin 30°$

$J_T = \frac{Mu\sqrt{3}}{2}$ $\quad v = \frac{u}{2}$

Que:) Repeat the previous problem for a movable ball instead of pin.







$(\angle IM_{M_2})$

$-J_T = Mv - Mu\cos 30° \quad —(1)$

$(\angle IM_{M_2})$ $\quad J_T = Mv \quad —(2)$

① + ②

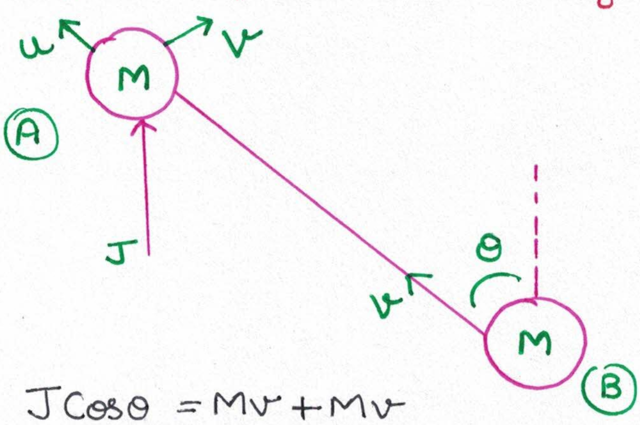
$$2Mv - Mu\cos 30^\circ = 0$$

$$v = \frac{u\sqrt{3}}{2}$$

$$v_B = \sqrt{\left(\frac{4\sqrt{3}}{4}\right)^2 + \left(\frac{4}{2}\right)^2}$$

$$= \frac{\sqrt{7}u}{4}$$

Que:) Find post impulse velocities for both the balls.



$$J\cos\theta = Mv + Mv$$

$$v = \frac{J\cos\theta}{2M} \quad \text{(velocity of ball B)}$$

$$J\sin\theta = Mv$$

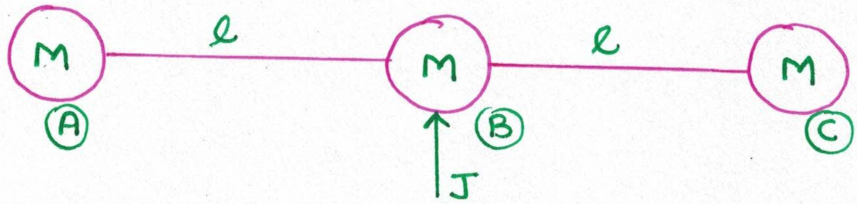
$$v = \frac{J\sin\theta}{M}$$

$$v_A = \frac{J}{M}\sqrt{\frac{\cos^2\theta}{4} + \sin^2\theta}$$

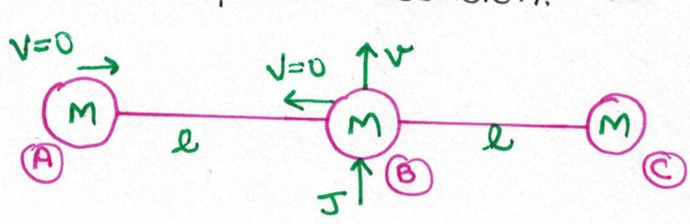
Que:) The system lies on a perfectly smooth surface, the Middle block is hit with a que stick.

(i) Will there be any impulsive tension.

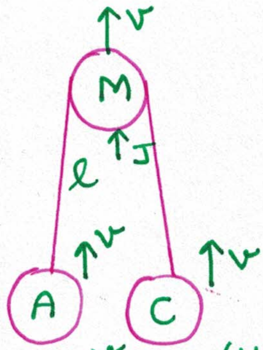
(ii) What will be the speed of ball B when A is going to collide with ©



(i) No, there is no impulsive tension.



(ii)



$\rightarrow v \leftarrow$ (Horizontal component)

$J = Mv + Mv + Mv$

$J = 3Mv$

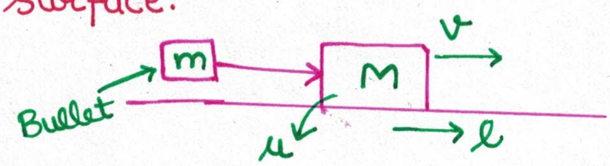
$v = \dfrac{J}{3M}$

as the system is on smooth horizontal plane, use COME

I.K.E = F.K.E

$$\underset{(B)}{\frac{1}{2}M\left(\frac{J}{M}\right)^2} + \underset{(A)}{0} + \underset{(C)}{0} = \frac{1}{2}m\left(\frac{J}{3M}\right)^2 + \frac{1}{2}m\left(\frac{J}{3M}\right)^2 + \underset{(B)}{v^2}$$

Que.) Find the distance travelled by the block on the rough surface.



<u>COM</u>

(friction is very small)

$mu = mv + Mv$

<u>CHIME</u>

$-umgl = 0 - \frac{1}{2}Mv^2$

impulse offered by bullet $= mu - mv$